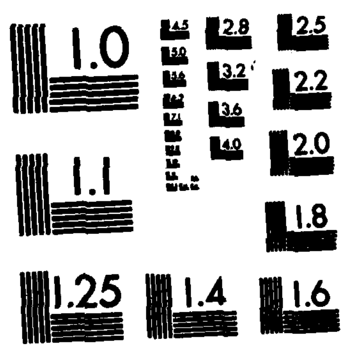
AD-A124 594 ASYMPTOTIC AND GEOMETRIC PROCEDURES FOR ESTIMATING CORRELATION FUNCTIONS F..(U) PENNSYLVANIA STATE UNIV UNIVERSITY PARK APPLIED RESEARCH LAB.. E L TITLEBAUM 1/1

UNCLASSIFIED 09 JUL 82 ARL/PSU/TM-82-151 F/G 12/1 NL

END
FILMED
DTIC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

6

ASYMPTOTIC AND GEOMETRIC PROCEDURES FOR ESTIMATING CORRELATION FUNCTIONS FOR FM SIGNALS

E. L. Titlebaum

Technical Memorandum
File No. TM 82-151
July 9, 1982
Contract No. N00024-79-C-6043

Copy No. 28

The Pennsylvania State University
Intercollege Research Programs and Facilities
APPLIED RESEARCH LABORATORY
Post Office Box 30
State College, PA 16801

Unlimited Distribution

NAVY DEPARTMENT

NAVAL SEA SYSTEMS COMMAND

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER TM 82-151	2. GOVT ACCESSION NO. A124 594
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle) ASYMPTOTIC AND GEOMETRIC PROCEDURES FOR ESTIMATING CORRELATION FUNCTIONS FOR FM SIGNALS	5. TYPE OF REPORT & PERIOD COVERED Interim
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) E. L. Titlebaum	8. CONTRACT OR GRANT NUMBER(s) N00024-79-C-6043
9. PERFORMING ORGANIZATION NAME AND ADDRESS Applied Research Laboratory P. O. Box 30 State College, PA 16801	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Naval Sea Systems Command, Code 63R-14 Department of the Navy Washington, DC 20362	12. REPORT DATE July 9, 1982
	13. NUMBER OF PAGES 33
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)	15. SECURITY CLASS. (of this report) UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited. Per Naval Sea Systems Command Sept. 9, 1982.

Accession For

Distribution/

Availability Codes

Avail and/or

Dist Special

A

DTIC COPY INSPECTED 2

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

FM, signal, correlation, function, estimation

(frequency modulation)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The first section of this paper uses the method of stationary phase in order to derive an asymptotic expression for the cross-correlation function for two FM signals whose instantaneous frequency curves intersect at one point in time-frequency space. Assuming slow variation in the amplitude and frequency modulation functions allows a simple geometric interpretation for the asymptotic result which is the square-root of an area measured in time-frequency space. This is shown in section two. The case for multiple intersections is discussed and shown, through an example, to depend on the relative

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

phases of the individual contributions. This intuitive method of evaluating correlation functions explains why V-chirp and SQFM signals have pedestals in their ambiguity functions.

Subject: Asymptotic and Geometric Procedures for Estimating Correlation Functions for FM Signals

References: See page 31.

ABSTRACT

The first section of this paper uses the method of stationary phase in order to derive an asymptotic expression for the cross-correlation function for two FM signals whose instantaneous frequency curves intersect at one point in time-frequency space. Assuming slow variation in the amplitude and frequency modulation functions allows a simple geometric interpretation for the asymptotic result which is the square-root of an area measured in time-frequency space. This is shown in section two. The case for multiple intersections is discussed and shown, through an example, to depend on the relative phases of the individual contributions. This intuitive method of evaluating correlation functions explains why V-chirp and SQFM signals have pedestals in their ambiguity functions.

TABLE OF CONTENTS

LIST OF FIGURES

I. Introduction.

Frequency Modulated (FM) signals with large time-bandwidth products are used in many areas, such as Sonar, Radar and Spread Spectrum Communications. (1,2) In order to take advantage of the pulse compression properties of these signals, many of these applications require coherent or matched filter processing as the appropriate receiver. The receiver response is usually expressed in terms of the ambiguity functions for the signals involved. Although there are many types of FM signals which have pulse compression properties, their ambiguity functions can have quite different properties. For example, linear FM signals (chirp) have a single ridge with no pedestal, whereas V-chirp FM signals have two small ridges near the time-frequency origin and an additional pedestal. Thus,it is desireable to have simple and intuitive methods for evaluating these ambiguity functions.

In the first part of this paper we use the method of stationary phase to derive an asymptotic formula for the crosscorrellation function of two FM signals whose instantaneous frequency curves cross at only one point in time. For this case, a simple geometric procedure is provided for estimating the crosscorrelation function, based upon the overlapping area of two templates generated in time-frequency space. For signals with more then one crossing in the time-frequency plane, the geometric procedure provides an approximate upper bound on the crosscorrealtion function, since the absolute phase shift between the two signals may not be known. In addition a predictable

oscillation in time occurs in the crosscorrelation function due to the changing phase between the components. Since the ambiguity function is a crosscorrelation function, we can apply these results directly to obtain the desired properties.

→ Sn 14

II. Asymptotic Theory.

In this section we consider the problem of asymptotically estimating the crosscorrelation function between two FM signal, based upon the crossings of their instantaneous frequency curves. We shall deal with complex analytic signals. In the appendix we establish the relationship between the crosscorrelation function of the real parts of the analytic signal and those for the analytic signals themselves.

We begin by defining instantaneous frequency. Suppose we have an analytic signal of the form

$$f(t) = a(t)\, e^{j\theta(t)} . \qquad (1)$$

By analytic we mean that the Fourier transform of f(t), F(w), satisfies the condition

$$F(\omega) = 0, \quad \omega < 0 . \qquad (2)$$

Thus the signal has a one-sided spectrum. The instantaneous frequency for f(t) is the time rate-of-change of its phase, thus

$$\omega_f(t) = \frac{d\theta(t)}{dt} . \qquad (3)$$

Hence, for example, if

$$\theta(t) = \frac{B}{2} t^2 + \omega_0 t \tag{4}$$

for a typical linear FM signal, then

$$\omega_f(t) = Bt + \omega_0 . \tag{5}$$

We assume that the two signals of interest are of the same form:

$$\begin{cases} f(t) = a(t)\, e^{jA\alpha(t)} & \text{(6-a)} \\ g(t) = b(t)\, e^{-jB\beta(t)} , & \text{(6-b)} \end{cases}$$

and the crosscorrelation is given by

$$R_{fg}(\tau) = \int_{-\infty}^{\infty} f(t)\, g^*(t+\tau)\, dt . \tag{7}$$

Without loss of generality we assume the analytic signals have unit energy. Thus

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |g(t)|^2 dt = 1 \tag{8}$$

By the Schwarz Inequality, the maximum values of their auto and crosscorrelation functions are unity. Their instantaneous frequencies are thus

$$\omega_f(t) = A \frac{d\alpha(t)}{dt} \tag{9-a}$$

and

$$\omega_g(t) = -B \frac{d\beta(t)}{dt} \tag{9-b}$$

The two constants , A and B, are assumed to be large.

Substituting equation (6) into equation (7) yields

$$R_{fg}(\tau) = \int_{-\infty}^{\infty} k(t,\tau)\, e^{j\chi h(t,\tau)}\, dt, \qquad (10)$$

where

$$\begin{cases} k(t,\tau) = a(t)\, b(t+\tau), & (11\text{-}a) \\ \chi\, h(t,\tau) = A\,\alpha(t) + B\,\beta(t+\tau). & (11\text{-}b) \end{cases}$$

If we assume that A > B, without loss of generality, then we have

$$\chi = A, \quad h(t,\tau) = \alpha(t) + \frac{B}{A}\beta(t). \qquad (12)$$

Applying the method of stationary phase[3] to the integral in equation (4) we have, as an asymptotic expression

$$R_{fg}(\tau) \sim \left[\frac{2\pi}{\chi \dfrac{d^2 h(\hat{t},\tau)}{dt^2}}\right]^{1/2} k(\hat{t},\tau)\, \exp\left[j\chi h(\hat{t},\tau) + j\pi/4\right], \qquad (13)$$

where the stationary point, $\hat{t}$, is the unique solution of the equation

$$w_f(\hat{t}) - w_g(\hat{t}) = 0.$$

It is assumed .hat g(t) is continuous and h(t) is twice

continuously differentiable with

$$\frac{\partial^2 h(\hat{t},\tau)}{\partial t^2} > 0 \quad .$$

Let us do an example of how this procedure can be applied. We will first consider the case of the crosscorrealation between two linear FM signals, one an up chirp and the other a down chirp. We let

$$\begin{cases} f(t) = a(t)\, e^{j\frac{A}{2}t^2}, \\ g(t) = b(t)\, e^{-j\frac{A}{2}(T-t)^2}, \\ a(t) = b(t) = \frac{1}{\sqrt{T}}, \quad 0 \le t \le T \end{cases}$$

The intantaneous frequency curves for the two signals are shown in Figure 1.

The unit energy condition reveals that

$$\int_0^T a^2(t)\, \partial t = \int_0^T b^2(t)\, \partial t = 1 \, .$$

Solving for the stationary point, with X = A, we have that

$$h(\hat{t},\tau) = \frac{t^2}{2} + \frac{(t+\tau-T)^2}{2} ,$$

and differentiating yields

$$\frac{\partial h(t,\tau)}{\partial t} = t + (t+\tau-T).$$

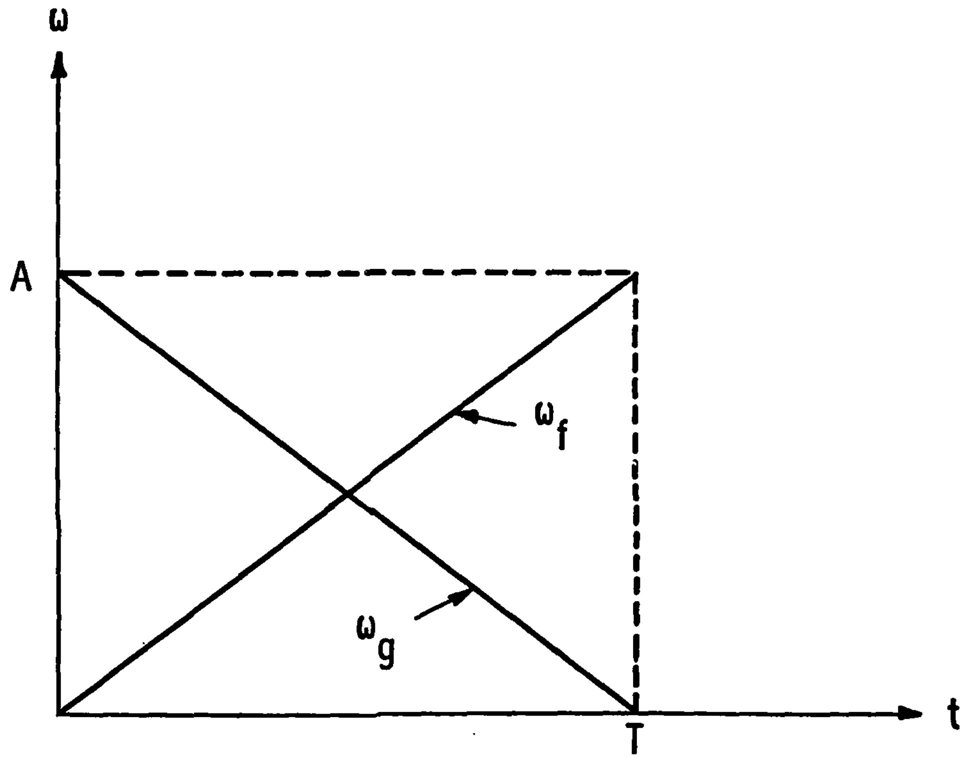


Figure 1. Instantaneous Frequency Plots for Linear FM Sweep Example.

Setting the derivative of h(t) equal to zero and solving for $\hat{t}$, we obtain

$$\hat{t} = \frac{T-\tau}{2}.$$

The second derivative condition is satisfied since

$$\frac{d^2 h(t,\tau)}{dt^2} = 2 > 0 ,$$

Solving for $k(\hat{t},\tau)$ and $h(\hat{t},\tau)$ we have

$$\begin{cases} k(\hat{t},\tau) = a\left(\frac{T-\tau}{2}\right) b\left(\frac{T-\tau}{2}\right), \\ \chi h(\hat{t},\tau) = \frac{A}{4}(\tau-T)^2 . \end{cases}$$

Thus we have as an asymptotic estimate of the crosscorrelation

$$R_{fg}(\tau) \sim a\left(\frac{T-\tau}{2}\right) b\left(\frac{T-\tau}{2}\right) \left[\frac{\pi}{A}\right]^{1/2} \exp\left[j\frac{A}{4}(\tau-T)^2 + j\frac{\pi}{4}\right].$$

If we further assume that the amplitudes have the form

$$a(t) = b(t) = \frac{1}{\sqrt{T}}, \quad 0 \le t \le T.$$

Thus the signals have rectangular envelopes. Then in the overlapping region $(-T < \tau < T)$ we have

$$|R_{fg}(\tau)| \sim \frac{1}{T}\left[\frac{\pi}{A}\right]^{1/2}$$

We observe that the total frequency deviation is AT radians/second or $2\pi F$ radians/second, then we obtain that

$$|R_{fg}(\tau)| \sim \frac{1}{T}\left[\frac{\pi T}{2\pi F}\right]^{1/2} = \frac{1}{\sqrt{2FT}}$$

We see that, as the time-bandwidth product gets large, the crosscorrelation function diminishes as the reciprocal square-root of the time-bandwidth product.

In order to check the asymptotic results, several computer examples were run. The signals were of the form

$$F(I) = A(I)\ EXP(BI^2 + WI), \quad I = 1,..,N.$$

The ampitudes were rectangular except for a 10% raised cosine at the ends of the signals. Figures 2. and 3. show the two signals for N = 128. The real and imaginary parts of the complex signals are plotted consecutively on each graph. The parameter are

a) Upsweep B = 0.003, W = 0.3

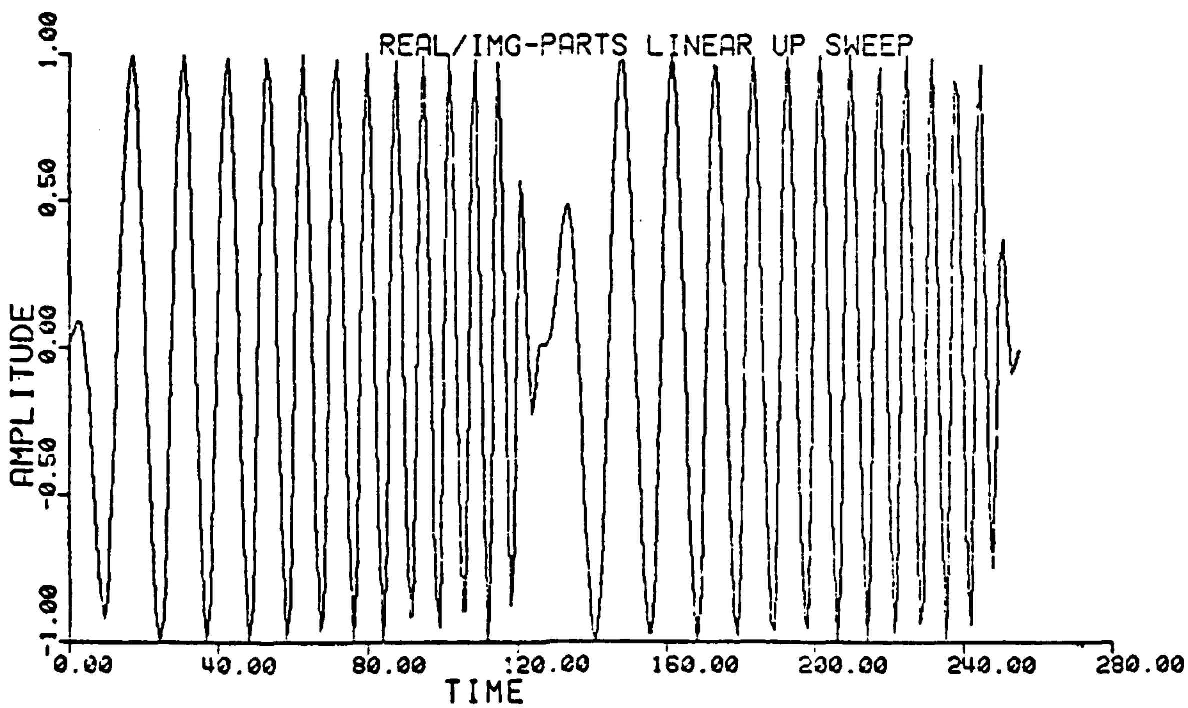
b) Downsweep B = -0.003, W = 1.068

Calculating the time duration and bandwidth we have

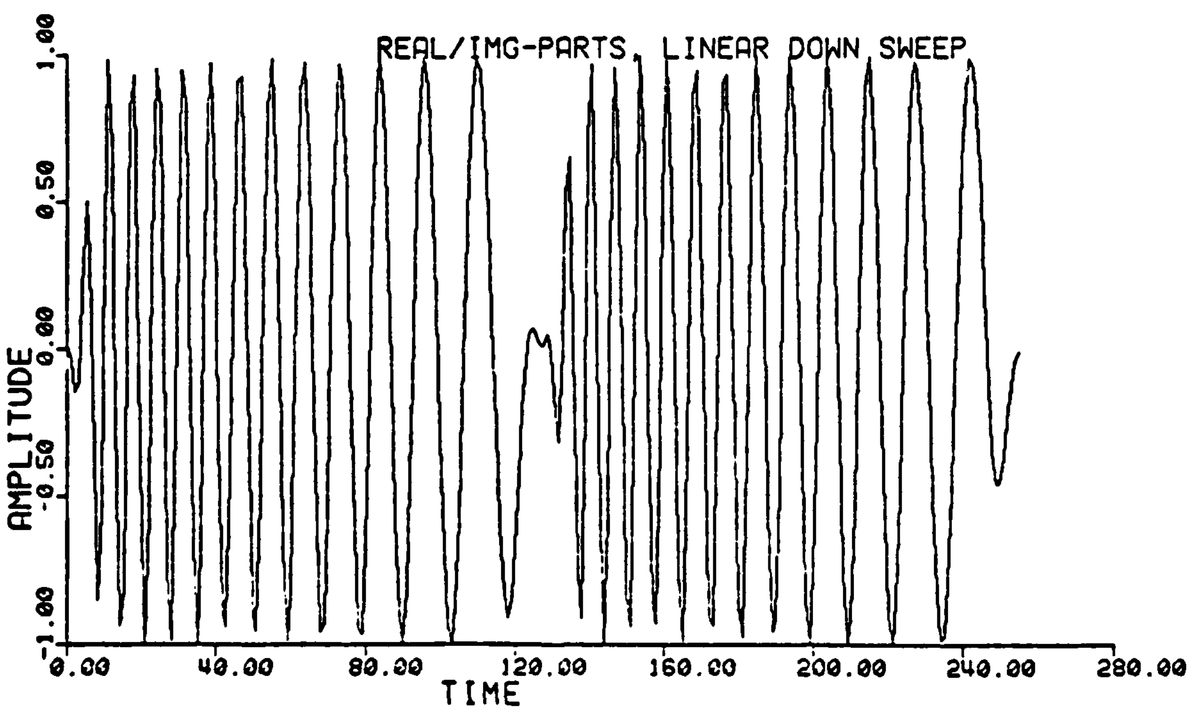
Time duration = (0.9)(128) = 115.2

Bandwidth = (0.9)(0.768/2) = 0.110

Figure 2. Real and Imaginary parts of 128 point Linear FM Upsweep signal.



<u>Figure 3.</u> Real and Imaginary parts of 128 point Linear FM Downsweep signal



Thus the crosscorrelation estimate is

$$|R_{fg}| \sim \frac{1}{\sqrt{2FT}} \qquad = 0.1986$$

Figure 4. shows the computed crosscorrelation function overlayed with the asymptotic estimate, shown as a staight line.

Suppose the 10% raised cosine window is not included then

Time duration = 128

Bandwidth = 0.768/2

and the crosscorrelation estimate is 0.179. The results of this case are shown in Figure 5., where the computed crosscorrelation function and the estimate or overlayed.

A seconds example with larger time-bandwidth product is given below. Here the parameters are

N = 256

a) Upsweep B = 0.003, W = 0.3

b) Downsweep B = -0.003, W = 1.836

Figure 6. shows the first half of the crosscorrelation functions of the signals overlayed with the asymptotic estimates. The results shown are for signals with and without the 10% raised cosine window.

Figure 4. Crosscorrelation function and Asymptotic Estimate of Linear FM Up and Downsweep with 10% raised cosine window. $\tau=0$ corresponds to Time Delay = 127.

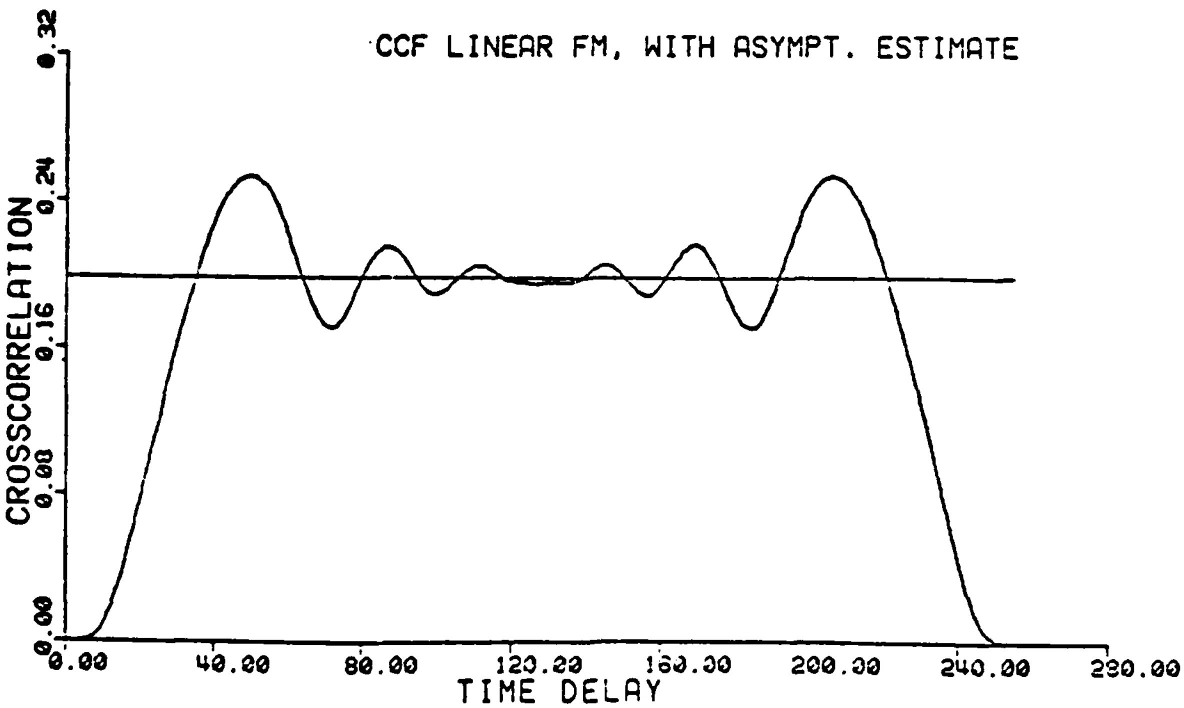


Figure 5. Crosscorrelation function and Asymptotic Estimate of Linear FM Up and Downsweep; NO window, τ=0 corresponds to Time Delay = 127.

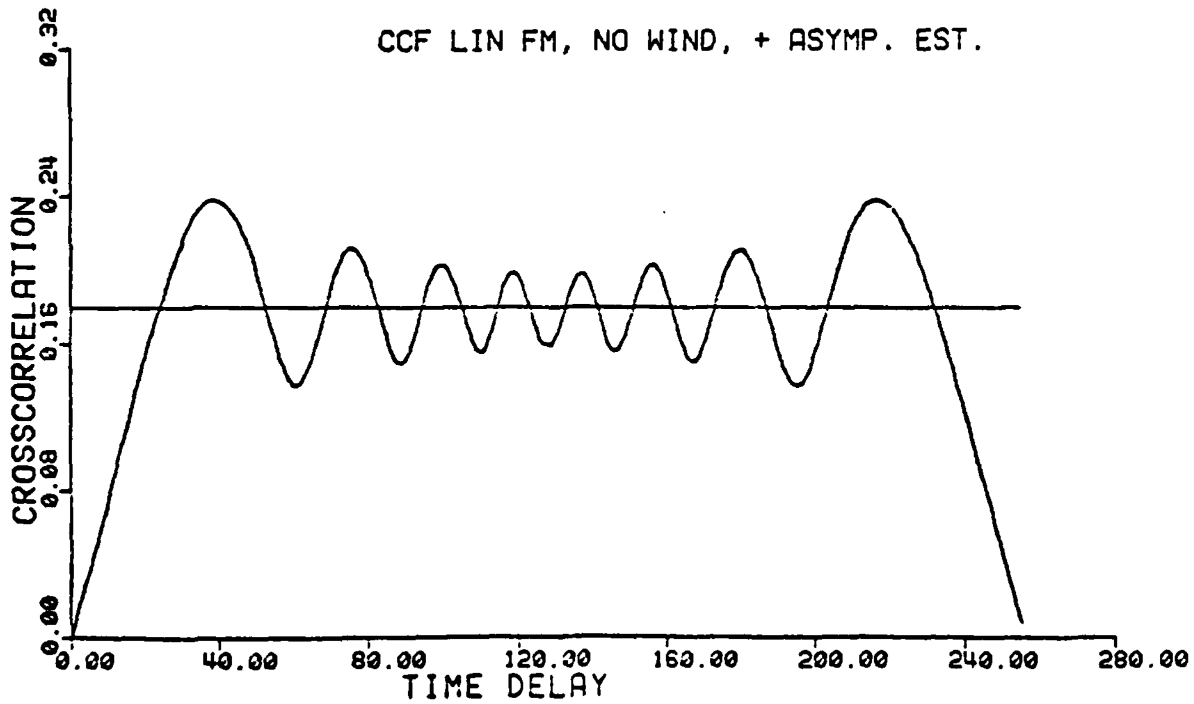
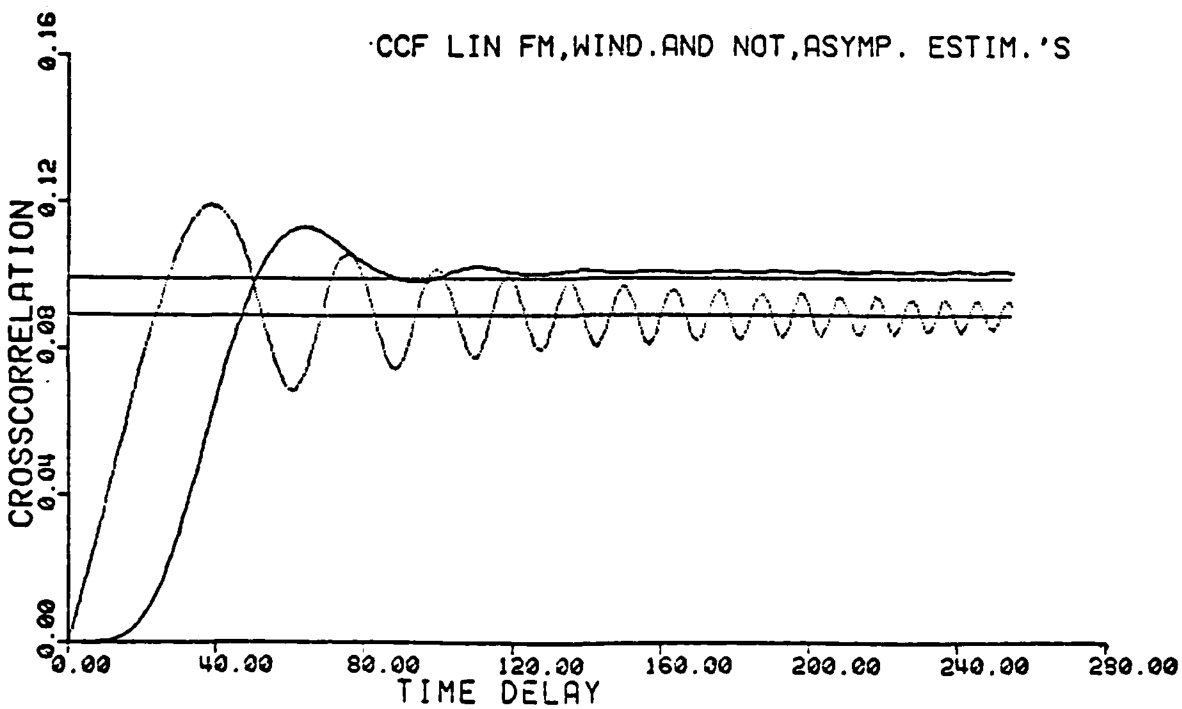


Figure 6. Half of crosscorrelation functions and Asymptotic estimates for 256 point Linear FM Up and Downsweeps, with and without 10% Raised cosine window.



III. Geometric Interpretation.

In this section we establish that a simple and intuitive geometric interpretation can be applied to the asymptotic results for the magnitude of the crosscorrelation function in equation (13). In particular we show that the magnitude of the right side of equation (13), the asymptotic estimate of the crosscorrelation function, is obtained directly from the area of a parallelogram in the time-frequency plane.

We begin by defining a parallelogram in terms of two sets of parallel lines in the time-frequency plane. Figure 7. shows the 4 lines. The area of the parallelogram is

$$\text{Area} = xy \sin(\theta). \qquad (15)$$

We must calculate the three quantities separately. To obtain sin( ) we observe that

$$\tan\alpha = A, \quad \tan\beta = B.$$

Note that w is defined with a negative slope. Thus we have that

$$\tan(90 - \alpha) = 1/A, \quad \tan(90 - \beta) = 1/B.$$

Taking inverse tangents and adding we have

$$\tan^{-1}(1/A) + \tan^{-1}(1/B) = 180 - \alpha - \beta = \theta.$$

July 9, 1982
ELT:cac

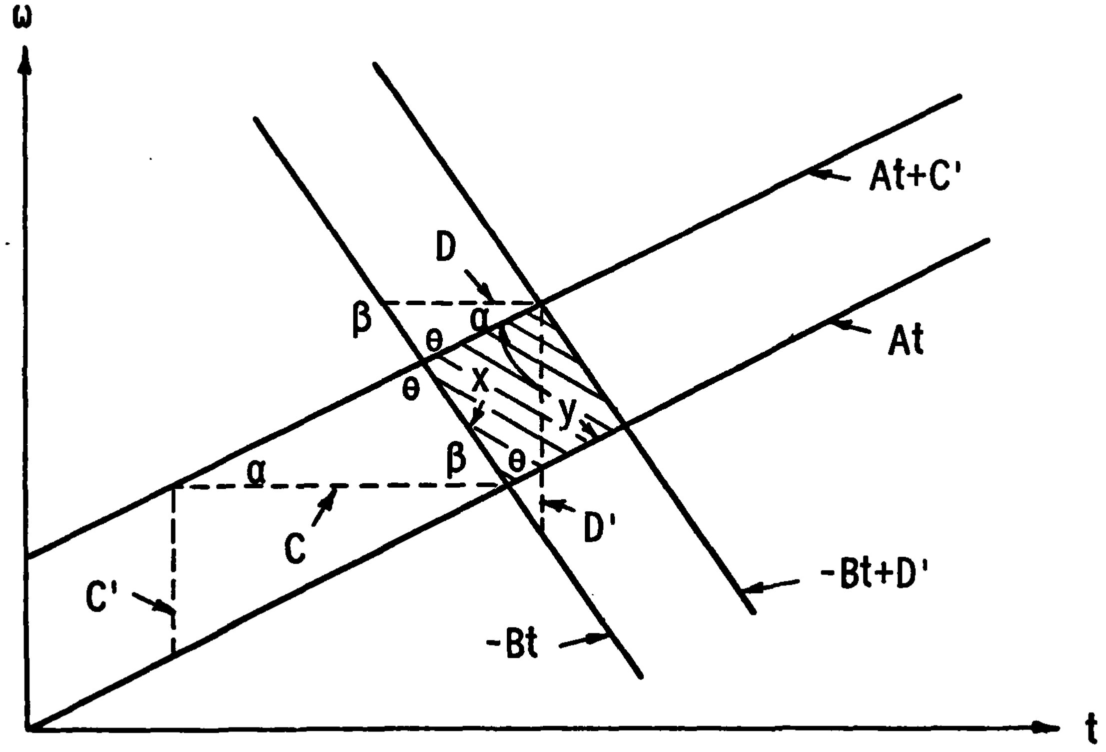


Figure 7. Geometry for Calculating the Area of Parallelogram Enclosed by Two Sets of Parallel Lines.

Thus

$$\tan\theta = \tan\left[\tan^{-1}(1/A) + \tan^{-1}(1/B)\right] = \frac{\frac{1}{A}+\frac{1}{B}}{1-\left(\frac{1}{A}\right)\left(\frac{1}{B}\right)}$$

$$= \frac{A+B}{AB-1},$$

and from this we obtain

$$\sin\theta = \frac{A+B}{\left[(A+B)^2+(AB-1)^2\right]^{1/2}} = \frac{A+B}{\left[(A^2+1)(B^2+1)\right]^{1/2}}. \quad (16)$$

To obtain x and y we recall that the diameter of the inscribed circle for a triangle is calculated as the ratio of any side to the sin of the opposite angle. Thus

$$\frac{C}{\sin\theta} = \frac{x}{\sin\alpha}, \quad \frac{D}{\sin\theta} = \frac{y}{\sin\beta}$$

or

$$x = \frac{C\sin\alpha}{\sin\theta}, \qquad y = \frac{D\sin\beta}{\sin\theta}, \qquad (17)$$

but

$$\sin\alpha = \frac{A}{(A^2+1)^{1/2}}, \quad \sin\beta = \frac{B}{(B^2+1)^{1/2}}. \quad (18)$$

Thus combining (16), (17), (18) and substituting into equation (15), we obtain that

$$\text{Area} = CD(AB)/(A+B), \tag{19}$$

Finally we observe that

$$\tan\alpha = A = C'/C, \quad \tan\beta = B = D'/D,$$

and substituiting we have for the area that

$$\text{Area} = C'D'/(A+B), \tag{20}$$

Here C' and D' are the respective vertical lengths between the two sets of parallel lines, and A and B are the tangents of the lines (one positive and one negative).

Consider now the squared magnitude of equation (13). We obtain

$$|R_{fg}(\tau)|^2 \sim \frac{2\pi}{\chi \dfrac{d^2 h(\hat{t},\tau)}{dt^2}} \cdot k^2(\hat{t},\tau) \tag{21}$$

which is

$$|R_{fg}(\tau)|^2 \sim \frac{2\pi}{\dfrac{d\omega_f(\hat{t},\tau)}{dt} - \dfrac{d\omega_g(\hat{t},\tau)}{dt}} \cdot a^2(\hat{t},\tau)\, b^2(\hat{t},\tau)\,. \tag{22}$$

If we now assume that the the two amplitude functions, a(t) and b(t), are slowly varying, and that the two instantaneous frequency functions are slowly varying, then we can associate the quantities in equation (20) with those in the asymptotic equation (22). Here the amplitudes are assumed to be constant, whereas the the frequency functions, w_f and w_g, are replaced by the first two terms of their Taylor series expansions about the point $\hat{t}$, for fixed τ. Then we may observe that

$$C' = a^2(\hat{t}, \tau) \quad , \quad D' = b^2(\hat{t}, \tau)$$

and

$$A = \frac{1}{2\pi} \frac{d\omega_f(\hat{t}, \tau)}{dt} \quad , \quad B = -\frac{1}{2\pi} \frac{d\omega_g(\hat{t}, \tau)}{dt} \; .$$

Then we have finally that

$$|R_{fg}(\tau)|^2 \sim [\text{Area}]^{1/2} .$$

and thus the crosscorrelation function can be obtained simply be taking the square-root of the parallelogram described above.

The results are true for the two linearly frequency modulated signa provided in the example, since all the assumptions, requiring slow variations, are true. Although we have not established these results when the slow variation assumptions are loosened, Several computer examples indicate that the results are still approximatly true, in those regions of τ for which the assumptions are not

true.

We can now see that, in order to obtain the asymptotic estimate of the crosscorrelation function, one need only construct a template for each signal, such as seen in Figure 8., and overlay the two templates for a fixed value of , and take the square-root of the reulting common area. Moving one template horizontally with respect to the other would yield a new value of crosscorrelation. Moving the same template vertically, simulating a narrowband Doppler shift, would provide a value of the narrowband cross-ambiguity function for the two signals. The time and frequency coordinates at which we are evaluating the crossambiguity function is the horizontal and vertical shifts respectively.

Although it may seem confusing as to why we plot the amplitude-squared along the frequency axis, i.e. $\omega' = a^{2}(\hat{t},\tau)$, observe that for the rectangulare amplitude case, as in the examples shown in the previous section, $a^{2}(\hat{t},\tau) = 1/T$, which has the units of frequency, so that the reults are, at least, consistent.

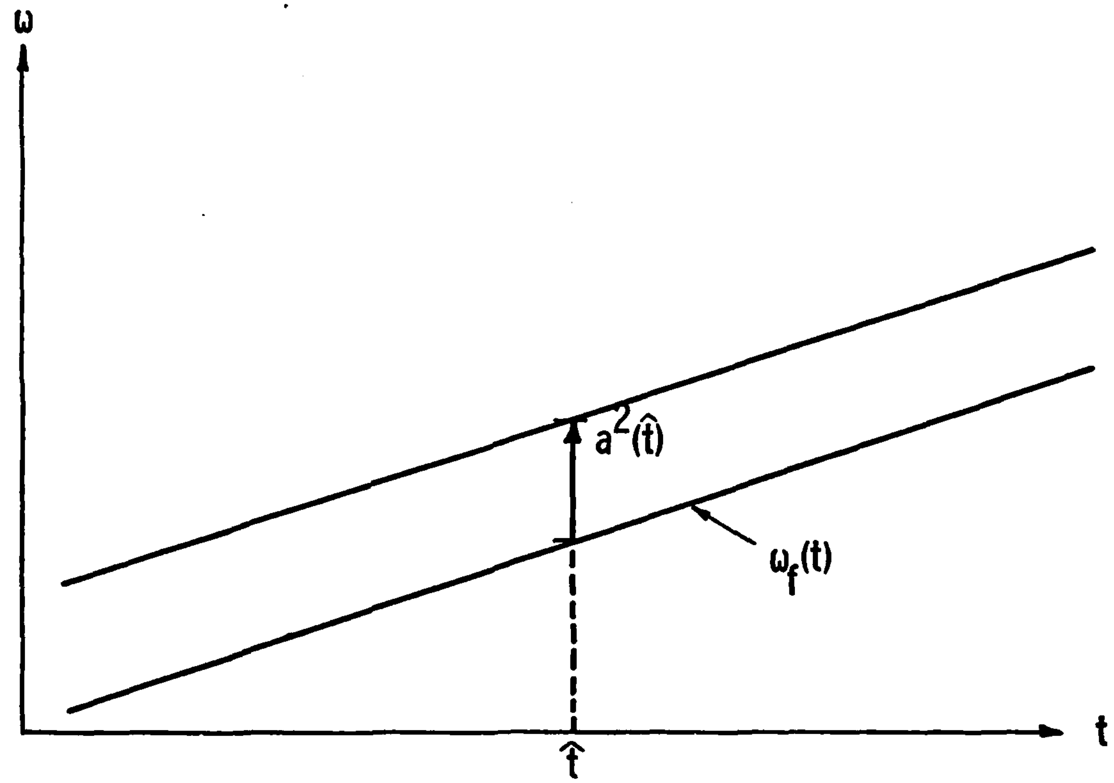


Figure 8. Template for a Typical Linear FM Upsweep Signal with Amplitude a(t).

IV. Multiple Intersections.

The case of multipe intersections of the instantaneous frequency curves for the two signals is considerably more complicated. The reason for this complexity becomes clear when we consider a simplified case. Suppose that for a particular delay, the instantaneous frequency curves have two intersections. Then we may calculate the crosscorrelation contribution from each intersection. Now the crosscorrelation function has two stationary points and, hence, has two complex terms which must be summed to obtain the complete result. In principle we have the asymptotic expression for the phase in equation (13) and could, thus, calculate the complex sum of the two contributions. This would only be possible if the two signals were specified analytically. However, if the signals were specified by their instantaneous frequency plots and amplitudes, then we do not have any knowledge of the absolute phase relationship between the two contributions. Since the addition of the contributions is complex, the resulting magnitude could have any value between the sum and difference of the individual contributions, depending upon their phase relationship.

An even further complication is that, the phase of each contibution is time-dependant. Thus the complex addition can change for each crosscorrelation estimate. This would be evidenced by an oscillation in the crosscorrelation function between the largest and smallest values of the complex addition.

In order to demonstrate this effect, Figure 9. shows the

crosscorrelation function for two signal with several intersecting regions. The signals are 128 points in duration with maximum frequency of 2π. Both signal have 10% raised cosine windows on each end. The first signal is the sum of two linear FM signals with frequency ranges (.5,1.5) and (2,3), while the second signal is a single linear FM sweep with range (3,1.25). If one constructs the two templates and passes them over one another horizontally then we observe that first there is no overlap, then a small region of one intersection, a region of two intersections, and finally a region of one intersection. The region of two intersections shows a rapid oscillation due to the relative phase, which is changing in time.

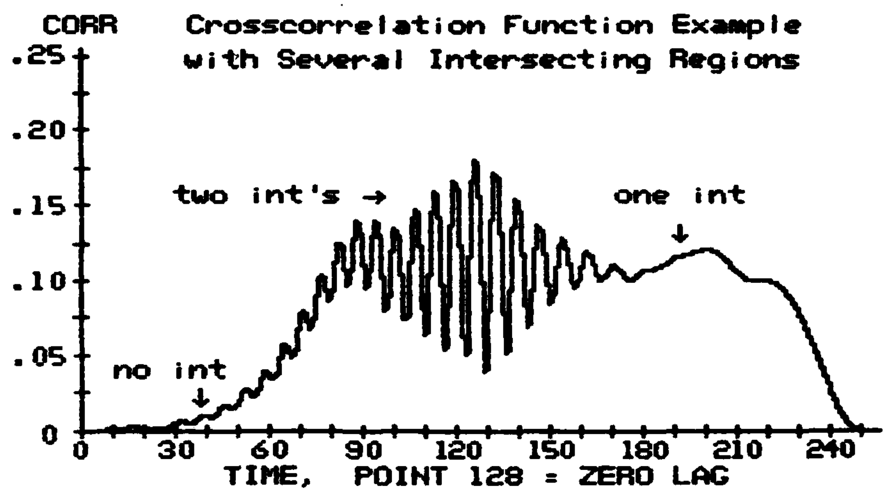


Figure 9. Crosscorrelation Function Example with Several Intersecting Regions.

V. Conclusions.

We have shown that a simple and intuitive geometric procedure can be used to obtain an asymptotic estimate of crosscorrelation functions for FM signals whose instantaneous frequency curves cross at only one point. In other cases this procedure provides an approximate upper bound on the behavior of the crosscorrelation function, in regions of more than one intersecting region. The procedure can be applied directly to narrowband ambiguity functions since the a Doppler shift is simpley a vertical shifting of the overlapping template. We can easily see reason why the autoambiguity function for a Vchirp or SQFM signal has a pronouced pedestal. Both of these signal pass through the same range of frequencies twice. Hence, for any shift, there will always be one intersection of the instantaneous frequency curves. For the Vchirp signal the pedestal is completly flat, since the parallelogram is the same area for all shifts. In the case of the SQFM signal the parallelogram reduces area as you move from the origin.

Perhaps the most useful aspect of this geometric procedure is the intuitive ability for quickly evaluating the properties of crosscorrelation and ambiguity functions of FM signals. Further, if one must evaluate the properties of many FM signals, a template need only be constructed for each one. Thus, one can evaluate these functions rather quickly.

Finally, as new electronic signal proccessing devices, which can represent signals in time-frequency space, become available,

the area concept may prove useful as a signal identifier or matched filter processor.

VI. References.

(1) Rihaczek, A. W., Principles of High-Resolution Radar, McGraw-Hill Book. Co., N.Y., 1969.

(2) Van Trees, H. L., Detection, Estimation, and Modulation Theory -- Part III, Wiley, N.Y. 1971.

(3) Erdelyi, A., Asymptotic Expansions, Dover Publications, 1956.

Appendix.

In this appendix we establish that if we have two analytic signals, k(t) and h(t), then the crosscorrelation of their real parts is equal to the real part of their crosscorrelation function. Further that the magnitude of the complex crosscorrelation function is the envelope of the real crosscorrelation.

To begin, we assume that $f_1(t)$ and $f_2(t)$ are real, unit energy signals, and the transform

$$\hat{f}(t) = H\left[f(t)\right] = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\tau)\,d\tau/(t-\tau) \qquad (A-1)$$

is the Hilbert transform. Then we form the analytic signals

$$\begin{cases} k(t) = \left[f_1(t) + j\hat{f}_1(t)\right]/\sqrt{2} & (A-2-a) \\ h(t) = \left[f_2(t) + j\hat{f}_2(t)\right]/\sqrt{2} & (A-2-b) \end{cases}$$

Denoting the inner product as

$$(k,h) = \int_{-\infty}^{\infty} k(t)h^*(t)\,dt \qquad (A-3)$$

and using the properties of Hilbert transforms and their spectra, it can be shown that if

$$F_1(w) = A_1(w)\,e^{j\theta_1(w)} \qquad (A-4-a)$$

$$F_2(w) = A_2(w)\, e^{j\theta_2(\omega)} \qquad \text{(A-4-b)}$$

then

$$(k,h) = (f_1, f_2) + j\left[\frac{1}{\pi}\int_0^{\infty} A_1(w)A_2(w)\sin(\alpha(w))\,dw\right] \qquad \text{(A-5)}$$

where

$$\alpha(w) = \theta_1(w) - \theta_2(w).$$

Recognizing that if

$$\begin{cases} f_1(t) = f(t) \\ f_2(t) = g(t+\tau) \end{cases}$$

Then the inner product is a crosscorrelation function and

$$R_e\left\{R_{kh}(\tau)\right\} = R_{fg}(\tau)$$

Further since has a one sided spectrum then its real and imaginary parts are themselves Hilbert transforms and hence $|R_{kh}(\tau)|$ is the envelope of $R_{fg}(\tau)$.

DISTRIBUTION LIST FOR ARL TM 82-151 by E. L. Titlebaum (University of Rochester), dated July 9, 1982

Commander
Naval Sea Systems Command
Department of the Navy
Washington, DC 20362

Attention: Dr. E. G. Liszka, PMS 406B
Copy No. 1

Attention: Mr. D. Porter, SEA 63R1
Copy No. 2

Attention: Mr. D. C. Houser, SEA 63R-14
Copy No. 3

Attention: Mr. C. D. Smith, SEA 63R
Copy No. 4

Attention: Mr. J. D. Antinucci, PMS 402
Copy No. 5

Attention: CAPT R. M. Wellborn, PMS 406
Copy No. 6

Attention: Mr. T. E. Douglas, PMS 406E1
Copy No. 7

Attention: Code SEA 9961 (Library)
Copies 8 and 9

Commanding Officer
Naval Underwater Systems Center
Newport, RI 02840

Attention: Dr. J. R. Short
Copy No. 10

Attention: Dr. J. M. D'Albora
Copy No. 11

Commander
Naval Air Systems Command
Department of the Navy
Washington, DC 20362

Attention: Mr. J. G. Smith
Copy No. 12

Director
Applied Research Laboratory
University of Texas
Austin, TX 78712

Attention: Dr. J. F. Willman
Copy No. 13

Marine Physical Laboratory
Scripps Institute of Oceanography
San Diego, CA 92152

Attention: W. S. Hodgkiss
Copy No. 14

Attention: K. M. Watson
Copy No. 15

Director
Applied Physics Laboratory
University of Washington
Seattle, WA 98105

Attention: Mr. C. Sienkiewicz
Copy No. 16

DARPA
525 Architect Building
1400 Wilson Boulevard
Arlington, VA 22209

Attention: Dr. Theodore Kooij
Copy No. 17

Commanding Officer
Naval Coastal Systems Center
Panama City, FL 32401

Attention: Dr. David Skinner
Copy No. 18

Bolt, Beranek and Newman, Inc.
1300 North 17th Street
Arlington, VA 22209

Attention: Dr. Henry Cox
Copy No. 19

Commander
Office of Naval Research
Department of the Navy
800 N. Quincy Street
Arlington, VA 22217

Attention: Statistics and
Probability Program
Code 411 SP
Copy No. 20

Defense Technical Information Center
Cameron Station
Alexandria, VA 22314

Copies No. 26 through 31

Department of Electrical Engineering
and Computer Science
Princeton University
Princeton, NJ 08450

Attention: Prof. Stuart C. Schwartz
Copy No. 21

Department of Electrical Engineering
Kelly Hall
University of Rhode Island
Kingston, RI 02881

Attention: Prof. Donald W. Tufts
Copy No. 22

ORNICON Corporation
3366 N. Torrey Pines Ct., Suite 320
La Jolla, CA 92037

Attention: Dr. Richard A. Altes
Copy No. 23

Systems Control Technology, Inc.
1801 Page Mill Road
P. O. Box 10180
Palo Alto, CA 94303

Attention: Dr. Benjamin Friedlander
Copy No. 24

Department of Electrical Engineering
College of Engineering and Applied Science
The University of Rochester
Rochester, NY 14627

Attention: Prof. E. L. Titlebaum
Copy No. 25